ABSTRACT

BLACKLEDGE, MICHAEL ALLAN. Closest Packing of Equal Spheres and Related Problems. (Under the direction of JOHN MONTGOMERY CLARKSON).

The two-dimensional packing problem is discussed, using the concept of the lattice, and the lattice which determines the closest packing of equal circles is presented. Also, closest packing in terms of density is discussed and the density value for the closest regular packing is derived.

The idea of sphere-clouds is introduced and used as an introduction to the closest packing of spheres. Lattice-like arrangements of spheres are considered, and the density of such a packing is determined.

Two proofs, one by John Leech and one by A. H. Boerdijk, are presented to show that it is impossible for thirteen spheres of equal radius to be in contact with a fourteenth sphere of the same radius.

A second related problem is presented, which when generalized reduces to the problem of finding the number of figures with (N + 1) vertices in N-space, choosing the vertices from given sets of points on given lines passing through a common point, subject to the restriction that no N lines lie in the same (N -1)-space. A solution by the author is presented and compared with a published solution.

CLOSEST PACKING OF EQUAL SPHERES
AND RELATED PROBLEMS

by

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1. INTRODUCTION

The problem of determining closest packing dates back at least as far as the argument in the marketplace of old of whether the customer was getting "good measure" in his purchase. St. Luke the evangelist writes in Chapter VI, Verse 38: "Give, and it shall be given unto you; good measure, pressed down, and shaken together, and running over, shall men give into your bosom. For with the same measure that ye mete withal it shall be measured to you again."

The purpose of this thesis is to determine the closest possible packing of equal spheres, and to investigate some related problems in this area. To accomplish this purpose, first the two-dimensional problem of determining the closest packing of circles in a plane is considered. Then the analogous problem of packing spheres in Euclidean three-space is investigated, using as a transition between the two problems the sphere cloud concept of L. Fejes Tóth.

Among the related problems is that of whether or not it is possible for thirteen spheres of equal size to be in contact with a fourteenth sphere. Two proofs are presented to show that such contact is indeed impossible, the first proof using the points of contact to form a network, and the second using the concept of central projection.

Finally, a problem is presented which generalizes to the idea of determining the number of (N + 1)-vertex figures in N-space, choosing the vertices from given points on given

intersecting lines, under the restriction that no N lines lie in the same (N - 1)-space. The author presents a solution and compares it with a published solution.

2. CLOSEST PACKING OF CIRCLES

We will consider one packing of circles to be closer than another if a (sufficiently large) prescribed region accomodates more circles of the first packing than of the second.

2.1 Circle Packings and Lattices

To determine the closest packing of circles, Hilbert and Cohn-Vossen (1952) use the idea of lattices. A square lattice is constructed by marking the four corners of a unit square in the plane. We then move the square one unit of length in the direction parallel to one of its sides, and mark the two new points indicated by the corners. We continue in this manner, and imagine the process to be repeated indefinitely, first in the original direction, and then in the opposite direction. Then we proceed in the directions orthogonal to our original directions, and thus cover the entire plane with points (see Figure 1). The totality of these points constitutes the square lattice, and it may be noted that instead of using a unit square to generate the lattice, any parallelogram that can be drawn on the lattice such that no lattice points are within its boundaries, and no lattice points lie on its boundaries except for vertices, may be used to generate the square lattice.

Now we will consider a special case of the general "unit lattices," that is, lattices that can be constructed from an arbitrary parallelogram of unit area in the manner in which we constructed the square lattice from the unit square. For any such lattices, the minimum distance $\underline{d}$ between any two

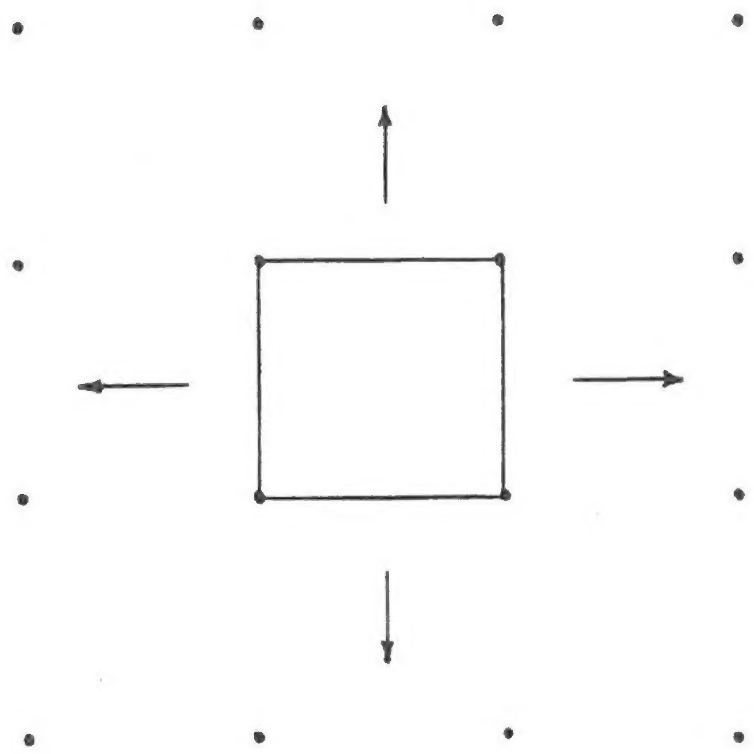

Figure 1. Construction of the square lattice

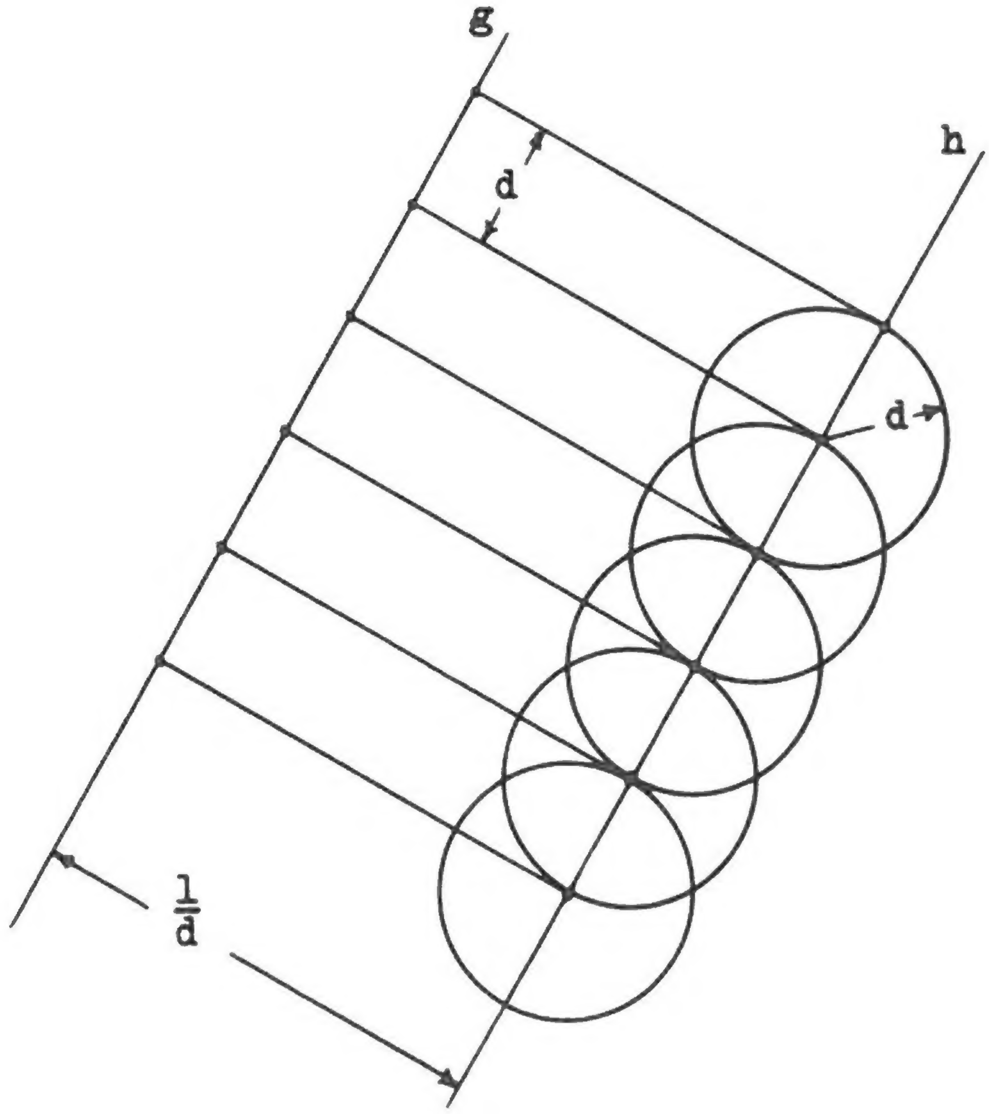

Figure 2. Construction of a unit lattice

lattice points is a characteristic quantity. d can be made arbitrarily small, as in the lattice generated by a rectangle with sides of length d and $\frac{1}{d}$. However, in order for us to be able to generate our lattice in the prescribed manner, d must have an upper limit. We shall determine this upper bound.

In any given unit lattice, let us choose any pair of lattice points separated by the minimum distance d. By definition of the lattice, there must be infinitely many more points of the lattice lying on the straight line g which passes through our two chosen points. By the unit property of the lattice, the straight line h parallel to g and at a distance $\frac{1}{d}$ from g must also pass through infinitely many lattice points, yet there are no lattice points between g and h. We now draw circles of radius d with centers at each of the lattice points on g (see Figure 2). The area covered by these circles is a strip of the plane bounded by circular arcs. Since every point within this strip is less than the radius d from a lattice point, none of these interior points is a lattice point itself, other than the centers. Thus we see that the shortest distance between our line g and the boundary of the strip is less than or, in the case where d = 1, equal to, $\frac{1}{d}$. This shortest distance is obviously the altitude of an equilateral triangle of side d, and thus

$$\frac{1}{d} \geq \frac{d}{2}\sqrt{3}$$

or

$$d \leq \sqrt{\frac{2}{\sqrt{3}}}$$

and we have found our upper bound for d, namely $\sqrt{\frac{2}{\sqrt{3}}}$. Exploring this idea of the equilateral triangle, we find that there actually is a lattice which attains this maximal value, namely the lattice generated by a parallelogram composed of two equilateral triangles (see Figure 3).

Any lattice of any area can be constructed from a unit lattice simply by increasing or decreasing the dimensions of the unit lattice. Thus if the generating parallelogram has an area of a^2 and if D is the minimum distance between two lattice points, the D is defined by

$$D = a\sqrt{\frac{2}{\sqrt{3}}}$$

Again the maximal distance is achieved if and only if the generating parallelogram is made up of two equilateral triangles.

It has been shown by Hilbert and Cohn Vossen (1952) that the area of large regions is asymptotically equal to the number of lattice points in the region multiplied by the area of the generating parallelogram. Thus for a given minimum distance between lattice points, the lattice built up of equilateral triangles not only has the smallest possible generating parallelogram, it also has the greatest number of points in a given large region.

Now let us construct circles with centers at the lattice points of this equilateral triangle lattice, with the radii of these circles being equal to one-half the minimum distance. Then none of these circles overlap, but tangencies occur. A

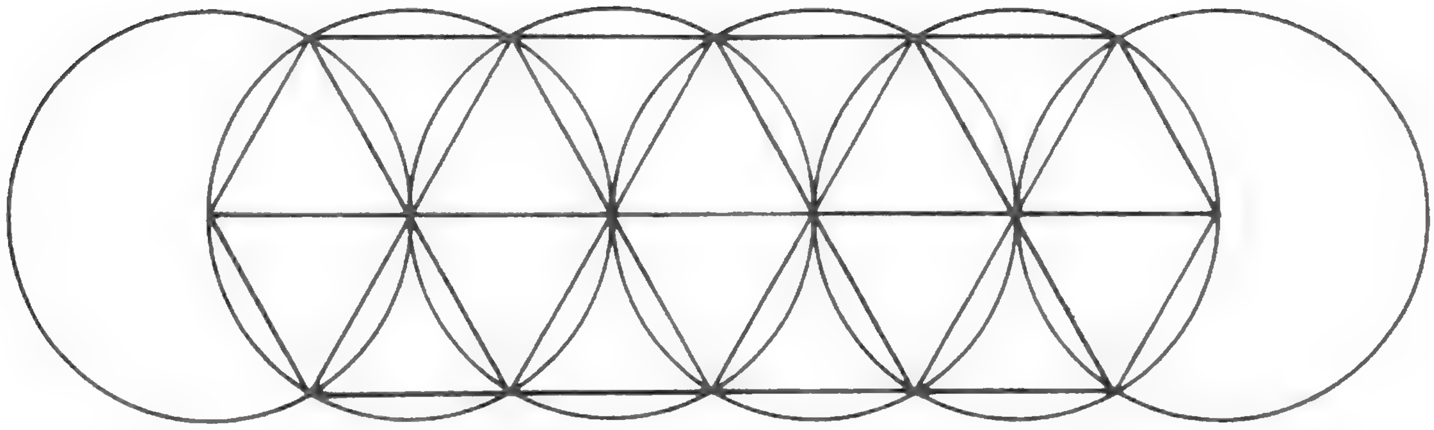

Figure 3. Lattice generated by equilateral triangle

system constructed in such a way is known as a regular packing of circles. Since, as we stated at the beginning of this section, we will consider one packing of circles to be closer than another if a (sufficiently large) prescribed region accomodates more circles of the first packing than of the other, we see that the lattice constructed on a basis of equilateral triangles gives us the closest packing of circles (see Figure 3).

2.2 Circle Packings and Density

Working with a more specific problem of the densest packing of circles, Segre and Mahler (1944) show that at most $\frac{A}{\sqrt{12}}$ circles of radius one can be placed in a convex polygon of area A, the polygon's angles not exceeding $\frac{2\pi}{3}$, such that no two circles overlap. Along these same lines, L. Fejes Tóth (1964) shows that a convex "hexagon," which he defines as a polygon with at most six sides and area A(H), contains n circles of equal radii without any overlapping, where

$$n = \frac{A(H)}{A(h)}$$

and A(h) is the area of the hexagon of least possible area circumscribed about a circle. Fejes Tóth also defines the density of a close packing by

$$\lim_{U\to\infty} \frac{n\,A(d)}{A(U)}$$

where A(d) is the area of a circle d, A(U) is the area of an arbitrary domain U, n is the maximal number of circles d which can be placed in U without "mutual overlapping," and "$U\to\infty$" denotes a limiting process by a continuous set of similarity transformations such that $A(U)\to\infty$.

Coxeter (1961) defines the density of the packing of equal circles to be the ratio of the area of a circle to the area of the polygon in which it is inscribed. Thus the closest packing will have the greatest such density, which must be less than unity. Furthermore, if the polygon circumscribing the circle has p sides each of length 2s, then its inradius is $r = s \cot \frac{\pi}{p}$ and its area is psr; thus, the density is

$$\frac{\text{Area of circle}}{\text{Area of polygon}} = \frac{\pi r^2}{psr} = \frac{\pi}{p}\frac{r}{s} = \frac{\pi}{p} \cot \frac{\pi}{p} = \frac{\frac{\pi}{p}}{\tan \frac{\pi}{p}}$$

Thus the density is an increasing function of p, and approaches one as p increases without limit. However, since the polygons must fit together in order to completely cover the plane, we restrict ourselves to using only regular polygons, and thus the only relevant values for the number of sides p are 3, 4, and 6. Hence the "best" value for p is 6, and the closest regular packing consists of the incircles of the faces of a regular hexagon, for which Hilbert and Cohn-Vossen (1952) get a density value of

$$\frac{\pi}{6} \cot \frac{\pi}{6} = \frac{\pi}{6}\sqrt{3} = \frac{\pi}{2\sqrt{3}} = 0.9069...$$

This procedure of packing circles as the incircles of regular polygons is an intuitive idea which can be derived from the packing of Figure 5 seeming obviously "more economical" than the packing of Figure 4.

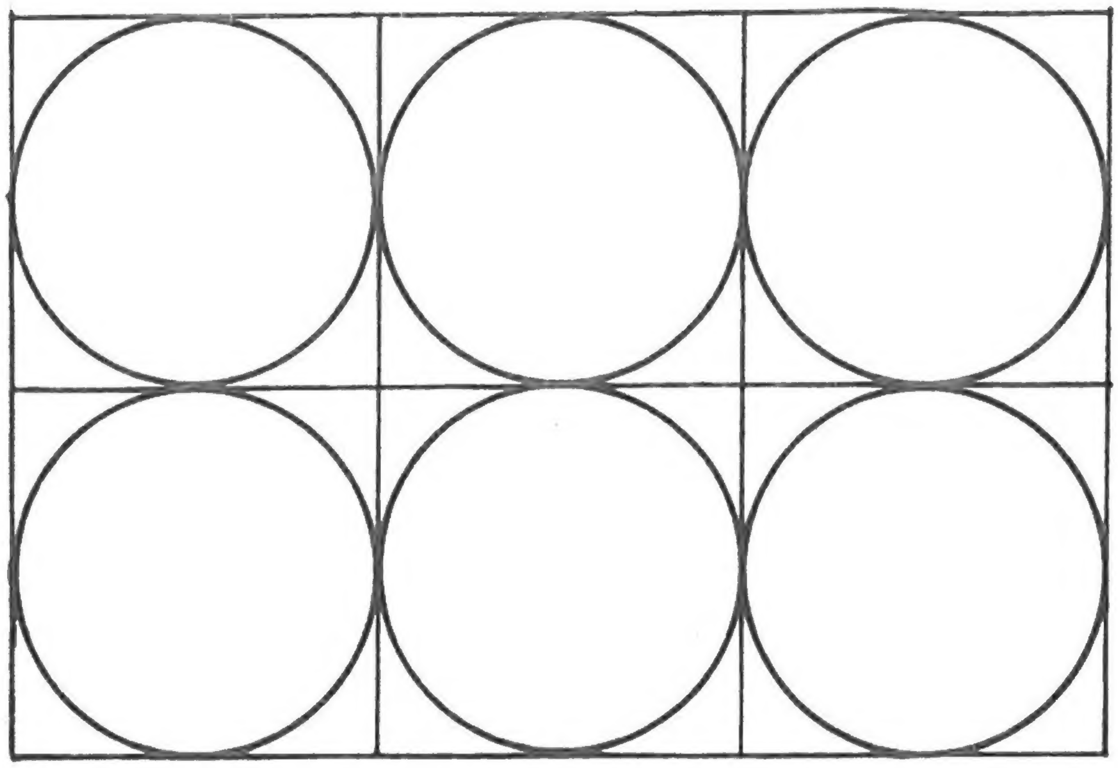

Figure 4. Packing circles as incircles of squares

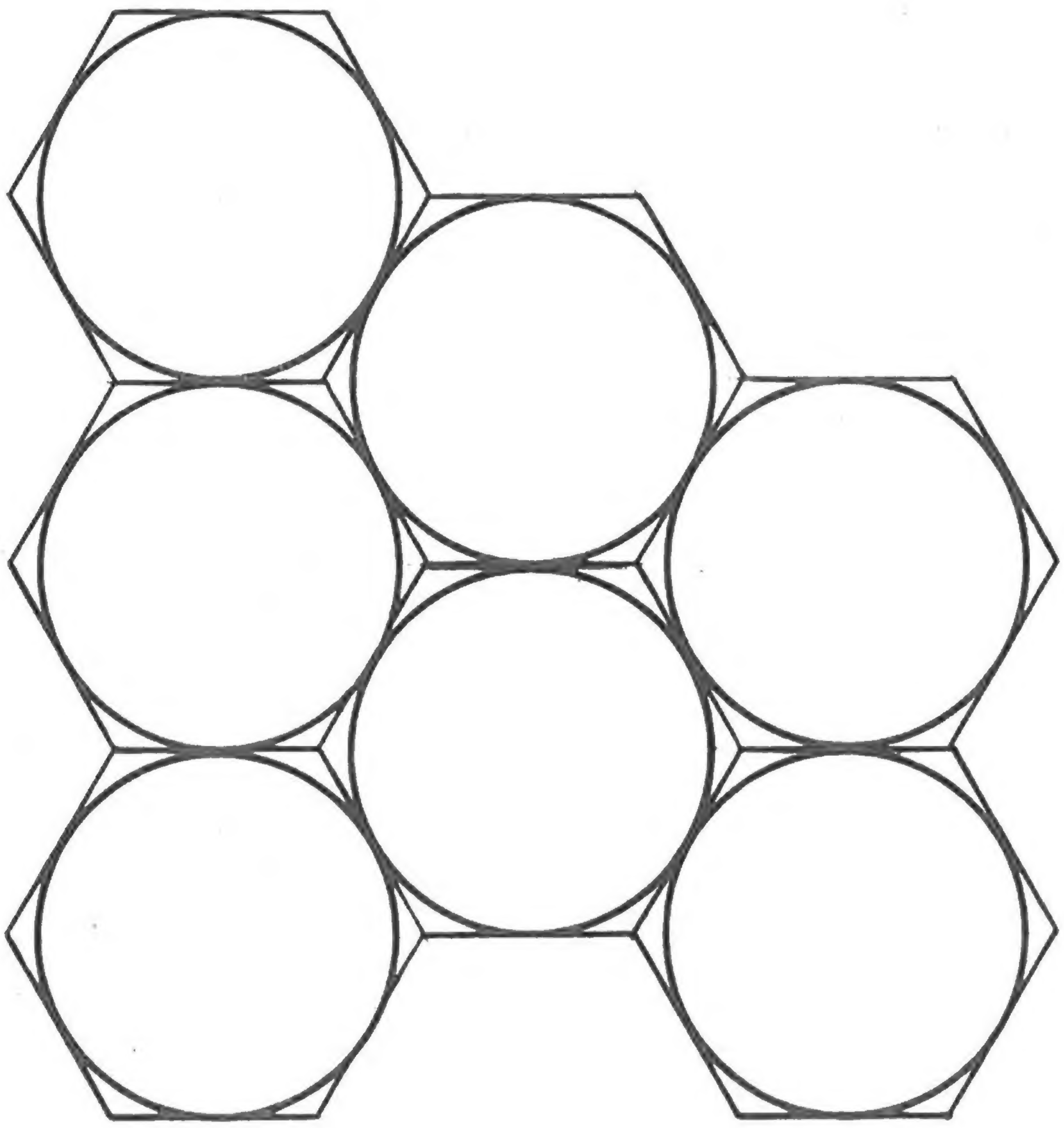

Figure 5. Packing circles as incircles of hexagons

Fejes Tóth (1964) notes that in 1938 Sinogowitz, making use of the researches of Sohncke, Barlow, Niggli, and others, divided the totality of the regular circle-packings into thirty-one classes, and exhibits these packings. Twenty-eight of these circle packings are stable in that each circle is fixed by its neighbors. By joining the centers of the circles in contact with each other by line segments, we obtain thirty-one tessellations of different types, including the eight semi-regular (Archimedean) tessellations having incongruent regular faces and equivalent vertices.